Electron–ion binary collisions in the presence of a magnetic field

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Binary collisions between ions and electrons in an external magnetic field are considered in second-order perturbation theory, starting from the unperturbed helical motion of the electrons. The calculations are done with the help of an improved BC which is uniformly valid for any strength of the magnetic field and the second-order energy and velocity transfers are treated in the interaction in Fourier space without specifying the interaction potential. The energy transfer is explicitly calculated for a regularized and screened potential which is both of finite range and less singular than the Coulomb interaction at the origin and as the limiting cases involves the Debye (i.e., screened) and Coulomb potentials. Two particular cases are considered in detaile: (i) Ion motion parallel to the magnetic field with an arbitrary strength. The energy transfer involves all harmonics of the electron cyclotron motion. (ii) The ion arbitrary motion with respect to the strong magnetic field when the electron cyclotron radius is much smaller than other characteristic length scales (e.g., screening length, pitch of electron helix etc.). In the latter case the energy transfer receives two contributions which are responsible for the electron guiding center and cyclotron orbit perturbations.

I. INTRODUCTION

In the presence of an external magnetic field **B** the problem of two charged particles cannot be solved in a closed form as the relative motion and the motion of the center of mass are coupled to each other. Therefore no theory exists for a solution of this problem that is uniformly valid for any strength of the magnetic field and the Coulomb force between the particles. The energy loss of ion beams and the related processes in a magnetized plasmas which are important in many areas of physics such as transport, heating, magnetic confinement of thermonuclear plasmas and astrophysics are examples of physical situations where this problem arises. Recent applications are the cooling of heavy ion beams by electrons [1, 2, 3] and the energy transfer for heavy-ion inertial confinement fusion (ICF) (see, e.g., [4] for an overview). The classical limit of a hydrogen or Rydberg atom in a strong magntic field also falls in this category (see, e.g., [5] and references therein) but in contrast to the free-free transitions (scattering) the total energy is negative there.

Numerical calculations have been performed for binary collisions (BC) between magnetized electrons [6, 7] and for collisions between magnetized electrons and ions [8, 9, 10]. In general the total energy W of the particles interacting in a magnetic field is conserved but the relative and center of mass energies are not conserved separetely. In addition, the presence of the magnetic field breaks the rotational symmetry of the system and as a consequence only the component of the angular momentum **L** parallel to the magnetic field $L_\parallel$ is a constant of motion. A different situation arises for the BC between an electron and heavy ion. As an ion is much heavier than an electron, its uniform motion is only weakly perturbed by collisions with the electrons and the magnetic field. In this case $L_\parallel$ is not conserved but there exists a conserved generalized energy K [10, 11] involving the energy of relative motion and a magnetic term. The seemingly simple problem of a charged particle interaction in a magnetic field is in fact a problem of considerable complexity and the additional degree of freedom of the cyclotron orbital motion produces a chaotic system with two degrees (or one degree for heavy ions) of freedom [12, 13, 14].

In this paper we consider the BC between electrons and heavy ion treating the interaction (Coulomb) with the ion as a perturbation to the helical motion of the magnetized electrons. This has been done previously in first order in the ion charge Z and for an ion at rest [15] and in up to $O(Z^2)$ for uniformly moving heavy ion [11, 16]. In Ref. [16] three regimes are identified, depending on the relative size of the parameters a (the cyclotron radius), s (the distance of the closest approach), and δ (the pitch of the helix). In earlier kinetic approaches [1, 2, 3] only two regimes have been distinguished: Fast collisions for $s < a$, where the Coulomb interaction is dominant and adiabatic collisions for $s > a$, where the magnetic field is important, as the electron performs many gyrations during the collision with the ion. The change ΔE_i of the energy of the ion has been related to the square of the momentum transfer Δp, which has been

calculated up to O(Z). This is somewhat unsatisfactory, as there is another O(Z^2) contribution to ΔE_i, in which the second-order momentum transfer enters linearly. Moreover, for applications in plasma physics (e.g., for calculation of the ion energy loss in a magnetized plasma) one calculates the angular avereged energy transfer which vanishes within first-order perturbation theory due to symmetry reasons and the ion energy change receives contribution only from higher orders [11]. Indeed, the transport phenomena, etc., are of order O(Z^2) in the ion charge.

In this paper we consider BC between ion and electrons in the presence of a magnetic field within the second order perturbation theory. The present paper is a continuation of our earlier study in Ref. [11] where the second-order energy transfer is calculated with the help of an improved BC treatment which is uniformly valid for any strength of the magnetic field and does not require the specification of the interaction potential. The paper is organized as follows. In Sec. II starting from the exact equation of motion of two charged particles moving in a magnetic field we discuss some basic results of the exact BC treatment for the energy and velocity transfers as well as the energy conservation. In the following Sec. III, we discuss the velocity and energy transfer during BC of magnetized electrons with ions for arbitrary magnetic fields and strengths of the electron-ion interaction potential. We assume that the ion mass M is much larger than the electron mass m. The equations of motion are solved in a perturbative manner up to the second order in Z starting from the unperturbed helical motion of the electrons in a magnetic field. Then in Secs. IV and V we turn to the explicit calculation of the ion second order energy transfer. For further applications (e.g., in cooling of ion beams) we consider the regularized and screened interaction potential which is both of finite range and less singular than the Coulomb interaction at the origin and as the limiting cases involves the Debye (i.e., screened) and Coulomb potentials. In Sec. IV the theory is applied to the energy transfer of an heavy ion moving parallel to the magnetic field **B** but without any restriction on **B**. The obtained energy transfer involves all cyclotron harmonics of the electron helical motion. The case of the strong magnetic field and arbitrary motion of the ion with respect to **B** is derived in Sec. V. It is shown that the energy transfer contains two terms which are responsible for the electron guiding center and cyclotron motion perturbations. In Sec. VI the results are summed up; some formulas for the second-order treatment are presented in the Appendices A and B.

II. BINARY COLLISION FORMULATION. GENERAL TREATMENT

Below we discuss the general equations of motion for two charged particles moving in a homogeneous magnetic field and the remaining conservation laws. From the velocity transfer we then proceed the energy transfer of particles during binary collision process. As shown in Ref. [11], the present treatment becomes more transparent in Fourier space.

A. Relative motion and conservation laws

We consider two point charges with masses m_1, m_2 and charges $q_1 e$, $q_2 e$, respectively, moving in a homogeneous magnetic field $\mathbf{B} = B\mathbf{b}$. We assume that the particles interact with the potential $q_1 q_2 \not{e}^2 U(\mathbf{r})$ with $\not{e}^2 = e^2/4\pi\varepsilon_0$, where ε_0 is the permittivity of the vacuum and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the relative coordinate of colliding particles. For charged particles the function $U(\mathbf{r})$ can be expressed, for instance, by the Coulomb potential, $U_{\mathrm{C}}(\mathbf{r}) = 1/r$. In plasma applications the infinite range of this potential is modified by the screening, e.g. $U_{\mathrm{D}}(\mathbf{r}) = e^{-r/\lambda}/r$ with a screening length λ which can be chosen as the Debye screening length λ_{D}, see, for example [17]. The quantum uncertainty principle prevents particles from falling into the center of these potentials. In a classical picture this can be achieved by regularization at the origin $U_{\mathrm{R}}(\mathbf{r}) = \left(1 - e^{-r/\not{\lambda}}\right) e^{-r/\lambda}/r$, see for example [18, 19], where $\not{\lambda}$ is a parameter, which may be related to the de Broglie wavelength.

In the presence of an external magnetic field, the Lagrangian and the corresponding equations of particles motion cannot, in general, be separated into parts describing the relative motion and the motion of the center of mass with velocities $\mathbf{v}$, $\mathbf{V}_{\mathrm{cm}}$ and coordinates $\mathbf{r}$, $\mathbf{R}_{\mathrm{cm}}$, respectively (see, e.g., [6]). Introducing the reduced mass $1/\mu = 1/m_1 + 1/m_2$ the equations of motion are

$$\dot{\mathbf{v}}(t) + \Omega_4 \left[\mathbf{v}(t) \times \mathbf{b}\right] = -\Omega_3 \left[\mathbf{V}_{\mathrm{cm}}(t) \times \mathbf{b}\right] + \frac{q_1 q_2 \not{e}^2}{\mu} \mathbf{F}\left(\mathbf{r}(t)\right), \tag{1}$$

$$\dot{\mathbf{V}}_{\mathrm{cm}}(t) - \Omega_1 \left[\mathbf{V}_{\mathrm{cm}}(t) \times \mathbf{b}\right] = -\Omega_2 \left[\mathbf{v}(t) \times \mathbf{b}\right], \tag{2}$$

where $q_1 q_2 \not{e}^2 \mathbf{F}\left(\mathbf{r}(t)\right)$ ($\mathbf{F} = -\partial U/\partial \mathbf{r}$) is the force exerted by the particle 2 on the particle 1. (The force which acts on the particle 2 is $-q_1 q_2 \not{e}^2 \mathbf{F}\left(\mathbf{r}(t)\right)$). The frequencies Ω_1, Ω_2, Ω_3 and Ω_4 are expressed in terms of the cyclotron

frequencies $\omega_{c1} = |q_1|eB/m_1$ and $\omega_{c2} = |q_2|eB/m_2$ of the particles 1 and 2, respectively,

$$\Omega_1 = \frac{m_1\varsigma_1\omega_{c1} + m_2\varsigma_2\omega_{c2}}{m_1 + m_2}, \quad \Omega_2 = \frac{\varsigma_2\omega_{c2} - \varsigma_1\omega_{c1}}{m_1 + m_2}\mu, \tag{3}$$

$$\Omega_3 = \varsigma_2\omega_{c2} - \varsigma_1\omega_{c1}, \quad \Omega_4 = -\mu\left(\frac{\varsigma_1\omega_{c1}}{m_1} + \frac{\varsigma_2\omega_{c2}}{m_2}\right). \tag{4}$$

Here $\varsigma_\nu = |q_\nu|/q_\nu$ with $\nu = 1, 2$. From Eqs. (1) and (2) follows the conservation of total energy

$$W = \frac{(m_1 + m_2)V_{\text{cm}}^2}{2} + \frac{\mu v^2}{2} + q_1 q_2 \ell^2 U(\mathbf{r}) = \text{const}, \tag{5}$$

but the relative and center of mass energies are not conserved separately.

The coupled, nonlinear differential equations (1) and (2) completely describe the motion of the particles. They have to be integrated numerically for a complete set of the initial conditions for solving the scattering problem. In the case of interaction of heavy ions ($m_2 = M$, $q_2 = Z$) with electrons ($m_1 = m$, $q_1 = -1$), i.e. $M \gg m$, the equations of motion can be further simplified, since $\mu \to m$, $\Omega_1, \Omega_2 \to 0$ and $\Omega_3, \Omega_4 \to \omega_c$ (see Eqs. (3) and (4)), where $\omega_c = eB/m$ is the cyclotron frequency of electrons. Equation (2) leads to $\mathbf{V}_{\text{cm}} \to \mathbf{v}_i = \text{const}$, where $\mathbf{v}_i$ is the heavy ion velocity, and Eq. (1) turns into

$$\dot{\mathbf{v}}(t) + \omega_c[\mathbf{v}(t) \times \mathbf{b}] = -\omega_c[\mathbf{v}_i \times \mathbf{b}] - \frac{Z\ell^2}{m}\mathbf{F}(\mathbf{r}(t)). \tag{6}$$

With the help of the equation of motion (6) it can be easily proven that the quantity

$$K = \frac{mv^2}{2} - Z\ell^2 U(\mathbf{r}) + m\omega_c\mathbf{r}[\mathbf{v}_i \times \mathbf{b}] \tag{7}$$

is a constant of motion. In contrast to the unmagnetized case, it thus follows that the relative energy transfer during ion-electron collision is proportional to $\delta r_\perp v_{i\perp}$, where $\delta r_\perp$ and $v_{i\perp}$ are the perpendicular components of the change of relative position and the ion velocity.

B.   Energy loss and velocity transfer

The rate at which the energy of an ion in a collision with an electron at time t changes is given by

$$\frac{dE_i(t)}{dt} = Z\ell^2\mathbf{v}_i \cdot \mathbf{F}(\mathbf{r}(t)), \tag{8}$$

as $Z\ell^2\mathbf{F}(\mathbf{r})$ is the force exerted by the electron on the ion. Integration with respect to time yields the energy transfer itself

$$\delta E_i(t) = Z\ell^2 \int_{-\infty}^{t} \mathbf{v}_i \cdot \mathbf{F}(\mathbf{r}(\tau))\, d\tau, \tag{9}$$

which after completion of the collision becomes

$$\Delta E_i = \delta E_i(t \to \infty) = Z\ell^2 \int_{-\infty}^{\infty} \mathbf{v}_i \cdot \mathbf{F}(\mathbf{r}(\tau))\, d\tau. \tag{10}$$

Here one has to integrate the equations of motion for the relative trajectories. The limit $M \gg m$ leading to Eq. (3) implies that the change in the ion energy is calculated under the assumption of a constant ion velocity. Alternatively this energy transfer can be expressed by the velocity transferred to the electrons during the collision. For that purpose we substitute $\mathbf{v}_i = \mathbf{v}_e(t) - \mathbf{v}(t)$ into Eq. (10) and split the electron velocity into two terms $\mathbf{v}_e(t) = \mathbf{v}_{e0}(t) + \delta\mathbf{v}(t)$, where $\mathbf{v}_{e0}(t)$ describes the helical motion in the magnetic field

$$\dot{\mathbf{v}}_{e0} + \omega_c[\mathbf{v}_{e0} \times \mathbf{b}] = 0 \tag{11}$$

and $\delta\mathbf{v}(t)$ the velocity transfer (we assume that $\delta\mathbf{v}(t) \to 0$ at $t \to -\infty$) due to the collision with the ion

$$\delta\dot{\mathbf{v}}(t) + \omega_c \left[\delta\mathbf{v}(t) \times \mathbf{b}\right] = -\frac{Z e^2}{m}\mathbf{F}\left(\mathbf{r}(t)\right). \tag{12}$$

This yields

$$\delta E_i(t) = Z e^2 \left[\int_{-\infty}^{t} \mathbf{v}_{e0}(\tau) \cdot \mathbf{F}\left(\mathbf{r}(\tau)\right) d\tau + \int_{-\infty}^{t} \delta\mathbf{v}(\tau) \cdot \mathbf{F}\left(\mathbf{r}(\tau)\right) d\tau + U\left(\mathbf{r}(t)\right)\right]. \tag{13}$$

In Eq. (13) we take into account that at $t \to -\infty$, $r(t) \to \infty$ and $U(\mathbf{r}(t)) \to 0$. The time integrals in Eq. (13) can be done with the help of the derivative of the scalar product $\mathbf{v}_{e0}(t) \cdot \delta\mathbf{v}(t)$. Using the equations of motion (11) and (12) we obtain

$$\frac{d}{dt}\left[\mathbf{v}_{e0}(t) \cdot \delta\mathbf{v}(t)\right] = -\frac{Z e^2}{m}\mathbf{v}_{e0}(t) \cdot \mathbf{F}\left(\mathbf{r}(t)\right), \tag{14}$$

which yields

$$Z e^2 \int_{-\infty}^{t} \mathbf{v}_{e0}(\tau) \cdot \mathbf{F}\left(\mathbf{r}(\tau)\right) d\tau = -m\mathbf{v}_{e0}(t) \cdot \delta\mathbf{v}(t). \tag{15}$$

Similarly from Eq. (12)

$$\frac{d}{dt}\left[\delta\mathbf{v}(t)\right]^2 = -\frac{2Z e^2}{m}\delta\mathbf{v}(t) \cdot \mathbf{F}\left(\mathbf{r}(t)\right) \tag{16}$$

which yields

$$Z e^2 \int_{-\infty}^{t} \delta\mathbf{v}(\tau) \cdot \mathbf{F}\left(\mathbf{r}(\tau)\right) d\tau = -\frac{m}{2}\left[\delta\mathbf{v}(t)\right]^2. \tag{17}$$

Thus

$$\delta E_i(t) = Z e^2 U\left(\mathbf{r}(t)\right) - m\mathbf{v}_{e0}(t) \cdot \delta\mathbf{v}(t) - \frac{m}{2}\left[\delta\mathbf{v}(t)\right]^2. \tag{18}$$

The last two terms in this equation represent the change in the electron energy due to the collision

$$\delta E_e(t) = \frac{m}{2}\left\{\left[\mathbf{v}_{e0}(t) + \delta\mathbf{v}(t)\right]^2 - \mathbf{v}_{e0}(t)\right\} = m\mathbf{v}_{e0}(t) \cdot \delta\mathbf{v}(t) + \frac{m}{2}\left[\delta\mathbf{v}(t)\right]^2. \tag{19}$$

This shows energy conservation

$$\delta E_i(t) + \delta E_e(t) - Z e^2 U(\mathbf{r}(t)) = 0. \tag{20}$$

As $U(\mathbf{r}(t \to \infty)) = 0$ the energy change of the ion can also be calculated from the velocity transfer $\Delta\mathbf{v} = \delta\mathbf{v}(t \to \infty)$ with the help of

$$\Delta E_i = -\delta E_e(t \to \infty) = -m\left(\mathbf{v}_{e0}(t) \cdot \Delta\mathbf{v} + \frac{1}{2}\Delta\mathbf{v}^2\right), \tag{21}$$

this method has been adopted in [16]. In this approach the potential $U(\mathbf{r})$ has to be specified at an early stage. In Sec. III we will show that Eq. (10) allows for a more general formulation in which the cut–off at large distances and the regularization at small distances can be treated easily.

Until now we have considered the energy transfer of an ion. In addition this energy transfer, ΔE_i, can be expressed by the change of the relative energy, ΔE_r, and the electron momentum transfer $\Delta\mathbf{p} = m\Delta\mathbf{v}$. Because we are dealing with heavy ion with $\mathbf{v}_i = $ const, the relative velocity transfer is the same as for the electrons, i.e. $\delta\mathbf{v}(t)$. Since the unperturbed relative velocity is $\mathbf{v}_0(t) = \mathbf{v}_{e0}(t) - \mathbf{v}_i$ (see Eq. (27) below) we can establish a simple relation between energy transfers ΔE_i and ΔE_r given by

$$\Delta E_i = -\Delta E_r - \mathbf{v}_i \cdot \Delta\mathbf{p}. \tag{22}$$

It is clear that the relative energy and momentum transfers depend only on the relative quantities and the ion velocity $\mathbf{v}_i$ is not explicitly involved in ΔE_r and $\Delta\mathbf{p}$. Thus, having the ion energy transfer the other quantities can be easily extracted from Eq. (22).

III. PERTURBATIVE TREATMENT. GENERAL THEORY

A. Trajectory correction

In this section we seek an approximate solution of Eq. (12) in which the interaction force between the ion and electrons is considered as a perturbation. Thus we have to look for the solution of Eq. (12) for the variables $\mathbf{r}$ and $\mathbf{v}$ in a perturbative manner

$$\mathbf{r}(t) = \mathbf{r}_0(t) + \mathbf{r}_1(t) + \mathbf{r}_2(t)..., \qquad \mathbf{v}(t) = \mathbf{v}_0(t) + \mathbf{v}_1(t) + \mathbf{v}_2(t)..., \tag{23}$$

where $\mathbf{r}_0(t), \mathbf{v}_0(t)$ are the unperturbed ion-electron relative coordinate and velocity, respectively, $\mathbf{r}_n(t), \mathbf{v}_n(t) \propto Z^n \mathbf{F}_{n-1}$ ($n = 1, 2, ...$) are the nth order perturbations of $\mathbf{r}(t)$ and $\mathbf{v}(t)$, which are proportional to Z^n. $\mathbf{F}_n(t)$ is the nth order correction to the ion-electron interaction force. Using the expansion (23) for the nth order corrections $\mathbf{F}_n$ we obtain

$$\mathbf{F}\left(\mathbf{r}(t)\right) = \mathbf{F}_0\left(\mathbf{r}_0(t)\right) + \mathbf{F}_1\left(\mathbf{r}_0(t), \mathbf{r}_1(t)\right) + ..., \tag{24}$$

where

$$\mathbf{F}_0\left(\mathbf{r}_0(t)\right) = \mathbf{F}\left(\mathbf{r}_0(t)\right) = -i \int d\mathbf{k}\, U(\mathbf{k})\mathbf{k} \exp\left[i\mathbf{k} \cdot \mathbf{r}_0(t)\right], \tag{25}$$

$$\mathbf{F}_1\left(\mathbf{r}_0(t), \mathbf{r}_1(t)\right) = \left(\mathbf{r}_1(t) \cdot \frac{\partial}{\partial \mathbf{r}}\right) \mathbf{F}(\mathbf{r})\bigg|_{\mathbf{r}=\mathbf{r}_0(t)} = \int d\mathbf{k}\, U(\mathbf{k})\mathbf{k}\left[\mathbf{k} \cdot \mathbf{r}_1(t)\right] \exp\left[i\mathbf{k} \cdot \mathbf{r}_0(t)\right]. \tag{26}$$

In Eqs. (25) and (26), we have introduced the ion-electron interaction potential $U(\mathbf{r})$ through $\mathbf{F}(\mathbf{r}) = -\partial U(\mathbf{r})/\partial \mathbf{r}$ and the force corrections have been written using a Fourier transformation in space.

We start with the zero-order unperturbed helical motion of the electrons. From Eq. (11) we obtain

$$\mathbf{v}_0(t) = \mathbf{v}_r + v_{e\perp}\left\{\mathbf{u}\cos(\omega_c t) + [\mathbf{b} \times \mathbf{u}]\sin(\omega_c t)\right\}, \tag{27}$$

$$\mathbf{r}_0(t) = \mathbf{R}_0 + \mathbf{v}_r t + a\left\{\mathbf{u}\sin(\omega_c t) - [\mathbf{b} \times \mathbf{u}]\cos(\omega_c t)\right\}, \tag{28}$$

where $\mathbf{u} = (\cos\varphi, \sin\varphi)$ is the unit vector perpendicular to the magnetic field, $v_{e\parallel}$ and $v_{e\perp}$ (with $v_{e\perp} \geq 0$) are the electron unperturbed velocity components parallel and perpendicular to $\mathbf{b}$, respectively, $\mathbf{v}_r = v_{e\parallel}\mathbf{b} - \mathbf{v}_i$ is the relative velocity of the electron guiding center, and $a = v_{e\perp}/\omega_c$ is the cyclotron radius. It should be noted that in Eqs. (27) and (28), the variables $\mathbf{u}$ and $\mathbf{R}_0$ are independent and are defined by the initial conditions.

The equation for the first-order velocity correction is given by

$$\dot{\mathbf{v}}_1(t) + \omega_c\left[\mathbf{v}_1(t) \times \mathbf{b}\right] = -\frac{Ze^2}{m}\mathbf{F}_0\left(\mathbf{r}_0(t)\right) \tag{29}$$

with the solutions

$$\mathbf{v}_1(t) = \frac{Ze^2}{m}\left\{-\mathbf{b}V_\parallel(t) + \mathrm{Re}\left[\mathbf{b}\left(\mathbf{b} \cdot \mathbf{V}_\perp(t)\right) - \mathbf{V}_\perp(t) + i\left[\mathbf{b} \times \mathbf{V}_\perp(t)\right]\right]\right\}, \tag{30}$$

$$\mathbf{r}_1(t) = \frac{Ze^2}{m}\left\{-\mathbf{b}P_\parallel(t) + \mathrm{Re}\left[\mathbf{b}\left(\mathbf{b} \cdot \mathbf{P}_\perp(t)\right) - \mathbf{P}_\perp(t) + i\left[\mathbf{b} \times \mathbf{P}_\perp(t)\right]\right]\right\}, \tag{31}$$

where we have introduced the following abbreviations

$$V_\parallel(t) = \int_{-\infty}^{t} d\tau\, \mathbf{b} \cdot \mathbf{F}_0\left(\mathbf{r}_0(\tau)\right), \quad \mathbf{V}_\perp(t) = e^{i\omega_c t}\int_{-\infty}^{t} d\tau\, e^{-i\omega_c \tau}\mathbf{F}_0\left(\mathbf{r}_0(\tau)\right), \tag{32}$$

$$P_\parallel(t) = \nu + \int_{-\infty}^{t} d\tau\,(t - \tau)\,\mathbf{b} \cdot \mathbf{F}_0\left(\mathbf{r}_0(\tau)\right), \quad \mathbf{P}_\perp(t) = \frac{1}{i\omega_c}\int_{-\infty}^{t} d\tau\left[e^{i\omega_c(t-\tau)} - 1\right]\mathbf{F}_0\left(\mathbf{r}_0(\tau)\right) \tag{33}$$

with $\nu = -tV_\parallel(t)\big|_{t\to-\infty}$ and have assumed that all corrections vanish at $t \to -\infty$. For instance, in the unscreened Coulomb case, the interaction force $\mathbf{F}_0$ must behave as $\mathbf{F}_0(\mathbf{r}_0(t)) \to 1/t^2$ for $|t| \to \infty$. Thus from Eq. (32) at $t \to \infty$ we obtain $V_\parallel(t) \to V_{0\parallel} = \text{const}$ and $\mathbf{V}_\perp(t) \to e^{i\omega_c t}\mathbf{V}_{0\perp}$, where $\mathbf{V}_{0\perp} = \text{const}$. The quantities $V_{0\parallel}$ and $\mathbf{V}_{0\perp}$ give the first order velocity correction in Eq. (30) after an electron-ion collision. In this limit, we find for the first order trajectory correction from Eqs. (32) and (33) $P_\parallel(t) = V_{0\parallel}t + P_{0\parallel}$, $\mathbf{P}_\perp(t) = -i(\mathbf{V}_{0\perp}/\omega_c)e^{i\omega_c t} + \tilde{\mathbf{P}}_{0\perp}$, where

$$P_{0\parallel} = \nu - \int_{-\infty}^{\infty} d\tau \tau \mathbf{b} \cdot \mathbf{F}_0(\mathbf{r}_0(\tau)), \quad \tilde{\mathbf{P}}_{0\perp} = \frac{i}{\omega_c}\int_{-\infty}^{\infty} d\tau \mathbf{F}_0(\mathbf{r}_0(\tau)). \tag{34}$$

For the Coulomb interaction $\nu = \mathbf{b}\cdot\mathbf{v}_r/v_r^3$ and $\nu = 0$ for any screened interaction potential. Note that for the Coulomb interaction the second term in the first relation of Eq. (34) tends to infinity (see, e.g., [11]). However the contribution of this term to the ion energy change vanishes after averaging over impact parameters.

Substituting Eqs. (25) and (28) into Eq. (33), and using the Fourier series of the exponential function $e^{iz\sin\theta}$ [20], we obtain for an arbitrary interaction potential

$$P_\parallel(t) = i\int d\mathbf{k}U(\mathbf{k})(\mathbf{k}\cdot\mathbf{b})e^{i\mathbf{k}\cdot\mathbf{R}_0}\sum_{n=-\infty}^{\infty}e^{in\psi}J_n(k_\perp a)\frac{e^{i\zeta_n(\mathbf{k})t}}{(\zeta_n(\mathbf{k})-i0)^2}, \tag{35}$$

$$\mathbf{P}_\perp(t) = i\int d\mathbf{k}U(\mathbf{k})\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{R}_0}\sum_{n=-\infty}^{\infty}e^{in\psi}J_n(k_\perp a)\frac{e^{i\zeta_n(\mathbf{k})t}}{(\zeta_n(\mathbf{k})-i0)(\zeta_{n-1}(\mathbf{k})-i0)}, \tag{36}$$

where J_n are the Bessel functions of the nth order. Here $\zeta_n(\mathbf{k}) = n\omega_c + \mathbf{k}\cdot\mathbf{v}_r$, $\psi = \varphi - \theta$, $\tan\theta = k_y/k_x$ and $k_\perp$ is the component of $\mathbf{k}$ transverse to the magnetic field. The quantities $V_\parallel(t)$ and $\mathbf{V}_\perp(t)$ are obtained directly from Eqs. (35) and (36) through the relations $V_\parallel(t) = \dot{P}_\parallel(t)$ and $\mathbf{V}_\perp(t) = \dot{\mathbf{P}}_\perp(t)$.

It should be noted that Eqs. (30) and (31) give formal but exact solutions for the velocity and trajectory corrections to the unperturbed quantities in Eqs. (27) and (28) if the first order force $\mathbf{F}_0(\mathbf{r}_0)$ in Eqs. (32) and (33) is replaced by the exact one, $\mathbf{F}(\mathbf{r})$ with exact relative coordinate $\mathbf{r}$. Substituting Eqs. (30) and (31) with exact force $\mathbf{F}(\mathbf{r})$ into Eq. (21) we obtain an exact relation for the ion energy transfer

$$
\begin{aligned}
\Delta E_i &= Ze^2\left\{\mathcal{V}_\parallel v_{e\parallel} + v_{e\perp}\left[(\mathbf{u}\cdot\mathbf{V}_c) - \mathbf{V}_s\cdot[\mathbf{u}\times\mathbf{b}]\right]\right\} \\
&+ \frac{Z^2e^4}{2m}\left\{2(\mathbf{V}_c\cdot[\mathbf{b}\times\mathbf{V}_s]) - \mathcal{V}_\parallel^2 - [\mathbf{b}\times\mathbf{V}_c]^2 - [\mathbf{b}\times\mathbf{V}_s]^2\right\}.
\end{aligned}
\tag{37}
$$

Here

$$\mathcal{V}_\parallel = \int_{-\infty}^{\infty} d\tau \mathbf{b}\cdot\mathbf{F}(\mathbf{r}(\tau)), \quad \left\{\begin{matrix}\mathbf{V}_s \\ \mathbf{V}_c\end{matrix}\right. = \int_{-\infty}^{\infty} d\tau \mathbf{F}(\mathbf{r}(\tau))\left\{\begin{matrix}\sin(\omega_c\tau) \\ \cos(\omega_c\tau)\end{matrix}\right. \tag{38}$$

are constants. The exact energy transfer (37) is now expressed by only the relative coordinate $\mathbf{r}(t)$.

B. First and second order energy transfers

The total energy change of the ion during an ion-electron collision is given by Eqs. (10). Insertion of Eq. (24) into the general expression (10) yields

$$\Delta E_i = \Delta E_i^{(1)} + \Delta E_i^{(2)} + ..., \tag{39}$$

where

$$\Delta E_i^{(1)} = Ze^2\int_{-\infty}^{\infty} dt\mathbf{v}_i\cdot\mathbf{F}_0(\mathbf{r}_0(t)), \quad \Delta E_i^{(2)} = Ze^2\int_{-\infty}^{\infty} dt\mathbf{v}_i\cdot\mathbf{F}_1(\mathbf{r}_0(t),\mathbf{r}_1(t)) \tag{40}$$

are the first- and second order energy transfer, respectively.

1. First order energy transfer

The first-order energy transfer can be obtained by substituting Eqs. (25) and (28) into the first one of Eqs. (40). This yields

$$\Delta E_i^{(1)} = -2\pi i Z \ell^2 \int d\mathbf{k} U(\mathbf{k}) (\mathbf{k} \cdot \mathbf{v}_i) e^{i\mathbf{k}\cdot\mathbf{R}_0} \sum_{n=-\infty}^{\infty} e^{in\psi} J_n(k_\perp a) \delta(\zeta_n(\mathbf{k})). \tag{41}$$

We now introduce the variable $\mathbf{s} = \mathbf{R}_{0\perp}^{(r)}$ which is the component of $\mathbf{R}_0$ perpendicular to the relative velocity vector $\mathbf{v}_r$. From Eqs. (27) and (28) we can see that $\mathbf{s}$ is the distance of closest approach for the guiding center of the electron helical motion. The energy loss is now given by the average of ΔE_i with respect to the initial phase of the electrons φ and the azimuthal angle of $\mathbf{s}$. For spherically symmetric interaction potentials ($U(\mathbf{r}) = U(r)$ and $U(\mathbf{k}) = U(k)$) the first order energy transfer gives no contribution due to symmetry and the ion energy change receives a contribution only from higher orders. In fact, Eq. (41) for the averaged first order energy change gives

$$\langle \Delta E_i^{(1)} \rangle = -2\pi i Z \ell^2 \int d\mathbf{k} U(k) (\mathbf{k} \cdot \mathbf{v}_i) J_0(\kappa s) J_0(k_\perp a) \delta(\mathbf{k} \cdot \mathbf{v}_r), \tag{42}$$

where $\kappa^2 = k^2 - (\mathbf{k} \cdot \mathbf{n}_r)^2$ and $\mathbf{n}_r = \mathbf{v}_r/v_r$. As the integrand is an odd function of $\mathbf{k}$ we have $\langle \Delta E_i^{(1)} \rangle = 0$.

2. Second order energy transfer

Inserting Eqs. (26), (28), (31), (35) and (36) into the second equation of Eqs. (40) one obtains

$$\Delta E_i^{(2)}(\mathbf{R}_0, \varphi) = \frac{\pi i Z^2 \ell^4}{m} \int d\mathbf{k} d\mathbf{k}' U(\mathbf{k}) U(\mathbf{k}') (\mathbf{k} \cdot \mathbf{v}_i) e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}_0} \tag{43}$$

$$\times \sum_{n;m=-\infty}^{+\infty} e^{in\psi + im\psi'} J_n(k_\perp a) J_m(k'_\perp a) \delta(\zeta_n(\mathbf{k}) + \zeta_m(\mathbf{k}')) G_m(\mathbf{k}, \mathbf{k}'),$$

where $\psi' = \varphi - \theta'$, and

$$G_m(\mathbf{k}, \mathbf{k}') = \frac{2g_1(\mathbf{k}, \mathbf{k}')}{(\zeta_m(\mathbf{k}') - i0)^2} + \frac{g_2(\mathbf{k}, \mathbf{k}') - ig_3(\mathbf{k}, \mathbf{k}')}{(\zeta_m(\mathbf{k}') - i0)(\zeta_{m-1}(\mathbf{k}') - i0)} \tag{44}$$

$$+ \frac{g_2(\mathbf{k}, \mathbf{k}') + ig_3(\mathbf{k}, \mathbf{k}')}{(\zeta_m(\mathbf{k}') - i0)(\zeta_{m+1}(\mathbf{k}') - i0)}$$

with

$$g_1(\mathbf{k}, \mathbf{k}') = -(\mathbf{k} \cdot \mathbf{b})(\mathbf{k}' \cdot \mathbf{b}), \quad g_2(\mathbf{k}, \mathbf{k}') = (\mathbf{k} \cdot \mathbf{b})(\mathbf{k}' \cdot \mathbf{b}) - (\mathbf{k} \cdot \mathbf{k}'), \tag{45}$$
$$g_3(\mathbf{k}, \mathbf{k}') = \mathbf{k} \cdot [\mathbf{k}' \times \mathbf{b}].$$

Next, for the practical applications, $\Delta E_i^{(2)}$ is averaged with respect to the initial phase of electrons φ and the azimuthal angle $\vartheta_\mathbf{s}$ of the impact parameter $\mathbf{s}$. After averaging the energy transfer $\Delta E_i^{(2)}$ with respect to φ the remaining part will depend on $\delta((\mathbf{k} + \mathbf{k}') \cdot \mathbf{v}_r)$, i.e. the component of $\mathbf{k} + \mathbf{k}'$ along the relative velocity $\mathbf{n}_r$. Thus this δ-function enforces $\mathbf{k}+\mathbf{k}'$ to lie in the plane transverse to $\mathbf{n}_r$ so that $e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}_0}\delta((\mathbf{k} + \mathbf{k}') \cdot \mathbf{v}_r) = e^{i\mathbf{Q}\cdot\mathbf{s}}\delta((\mathbf{k} + \mathbf{k}') \cdot \mathbf{v}_r)$, where $\mathbf{Q} = \mathbf{k}_\perp^{(r)} + \mathbf{k}'^{(r)}_\perp$ and $\mathbf{k}_\perp^{(r)}$ is the component of $\mathbf{k}$ transverse to $\mathbf{n}_r$. The result of the angular averaging reads

$$\langle \Delta E_i^{(2)} \rangle_{\varphi, \vartheta_\mathbf{s}} = \frac{\pi i Z^2 \ell^4}{m} \int d\mathbf{k} d\mathbf{k}' U(\mathbf{k}) U(\mathbf{k}') (\mathbf{k} \cdot \mathbf{v}_i) J_0(Qs) \delta((\mathbf{k} + \mathbf{k}') \cdot \mathbf{v}_r) \tag{46}$$

$$\times \sum_{n=-\infty}^{\infty} (-1)^n e^{in(\theta - \theta')} J_n(k_\perp a) J_n(k'_\perp a) G_n(\mathbf{k}, \mathbf{k}').$$

This series representation of the second-order energy transfer is valid for any strength of the magnetic field. Besides the direction $\mathbf{b}$ of the magnetic field and the direction $\mathbf{n}_r = \mathbf{v}_r/v_r$ of the relative velocity is singled out in the argument

of the δ-function and summand of the n-summation. This prevents a closed evaluation of the energy transfer (46). However the limiting case of an ion motion parallel to a magnetic field of arbitrary strength and the case of an arbitrary motion in a strong magnetic field can be treated in a straightforward manner.

The calculation of the angular-averaged relative energy and momentum transfers is performed by the similar method as for deriving Eq. (46). Note, however, that due to the symmetry reason the transverse part of the momentum transfer vanishes, $\mathbf{v}_{i\perp} \cdot \langle \Delta \mathbf{p}_\perp \rangle = 0$, and Eq. (22) is reduced to $\langle \Delta E_i^{(2)} \rangle = -\langle \Delta E_r^{(2)} \rangle - v_{i\parallel} \langle \Delta p_\parallel^{(2)} \rangle$.

It is also usefull to integrate the $\varphi, \vartheta_{\mathbf{s}}$-averaged ion energy change, $\langle \Delta E_i^{(2)} \rangle_{\varphi,\vartheta_{\mathbf{s}}}$, over the impact parameters s in the full 2D space. Thus we can introduce a generalize cross section [9, 11] through the relation

$$
\sigma(\mathbf{v}_r, \mathbf{v}_i) = 2\pi \int_0^\infty \langle \Delta E_i^{(2)} \rangle_{\varphi,\vartheta_{\mathbf{s}}} s\, ds = -\frac{(2\pi)^4 Z^2 e^4}{2mv_r} \int d\mathbf{k} |U(\mathbf{k})|^2 (\mathbf{k} \cdot \mathbf{v}_i) \tag{47}
$$
$$
\times \sum_{n=-\infty}^\infty J_n^2(k_\perp a) \left\{ k_\parallel^2 \delta'(\zeta_n(\mathbf{k})) + \frac{k_\perp^2}{2\omega_c} [\delta(\zeta_{n+1}(\mathbf{k})) - \delta(\zeta_{n-1}(\mathbf{k}))] \right\},
$$

where $\delta'(x)$ defines the derivative of the δ-function with respect to the argument. Note that for the Coulomb interaction $U(k) = U_{\mathrm{C}}(k)$, the full 2D integration over the $\mathbf{s}$-space results in a logarithmic divergence of the $\mathbf{k}$-integration in Eq. (47). This will be explicitly shown in the next section. To cure this, we should introduce the cutoff parameters $k_{\min}$ and $k_{\max}$, see [11] for details.

IV. ION PARALLEL MOTION AND ARBITRARY MAGNETIC FIELD

The averaged energy transfer, Eq. (46), can be evaluated without further approximation for an ion motion parallel to the magnetic field and assuming axially symmetric interaction potential, $U(\mathbf{k}) = U(|k_\parallel|, k_\perp)$. In this case the averaged energy transfer can be represented as the sum of all cyclotron harmonics. For the parallel motion of the ion, $\mathbf{v}_{i\perp} = 0$ and $\mathbf{v}_r = (v_{e\parallel} - v_{i\parallel}) \mathbf{b} = v_{r\parallel} \mathbf{b}$, where $v_{i\parallel}$ and $\mathbf{v}_{i\perp}$ are the components of ion velocity along and transverse to the magnetic field, respectively. In general case setting $\mathbf{v}_{i\perp} = 0$, we have from Eq. (46)

$$
\langle \Delta E_{i\parallel}^{(2)} \rangle = \frac{\pi i Z^2 e^4 v_{i\parallel}}{m|v_{r\parallel}|} \int d\mathbf{k} d\mathbf{k}' U(\mathbf{k}) U(\mathbf{k}') k_\parallel J_0(|\mathbf{k}'_\perp + \mathbf{k}_\perp|s) \delta(k'_\parallel + k_\parallel) \tag{48}
$$
$$
\times \sum_{n=-\infty}^\infty (-1)^n e^{in(\theta-\theta')} J_n(k_\perp a) J_n(k'_\perp a) G_n(\mathbf{k}, \mathbf{k}').
$$

Introducing cylindrical coordinates along $\mathbf{b}$ and using the addition theorem for the Bessel functions $J_0(|\mathbf{k}'_\perp + \mathbf{k}_\perp|s)$ [20] the energy transfer (48) finally reads

$$
\langle \Delta E_{i\parallel}^{(2)} \rangle = \frac{4Z^2 e^4 v_{i\parallel}}{m\delta^2 v_{r\parallel}^3} \sum_{n=1}^\infty n^2 \left\{ 3U_n(k_\parallel, a, s) + k_\parallel \frac{\partial}{\partial k_\parallel} U_n(k_\parallel, a, s) \right. \tag{49}
$$
$$
\left. + \frac{\delta^2}{2n} [V_n(k_\parallel, s, a) - V_{n+1}(k_\parallel, s, a) + V_n(k_\parallel, a, s) - V_{n+1}(k_\parallel, a, s)] \right\}_{k_\parallel = n/\delta},
$$

where $\delta = |v_{r\parallel}|/\omega_c$ is the pitch of the electron helix, divided by 2π, and

$$
U_n(k_\parallel, a, s) = \left[\frac{(2\pi)^2}{2} \int_0^\infty U(k_\parallel, k_\perp) J_n(k_\perp a) J_n(k_\perp s) k_\perp dk_\perp \right]^2, \tag{50}
$$

$$
V_n(k_\parallel, a, s) = \left[\frac{(2\pi)^2}{2} \int_0^\infty U(k_\parallel, k_\perp) J_n(k_\perp a) J_{n-1}(k_\perp s) k_\perp^2 dk_\perp \right]^2. \tag{51}
$$

We recall that for the ion parallel motion the last term in Eq. (7) vanishes, i.e. the relative energy is conserved and $\langle \Delta E_r^{(2)} \rangle = 0$. Therefore, the relation between ion energy and relative momentum transfers is simplified to $\langle \Delta E_i^{(2)} \rangle = -v_{i\parallel} \langle \Delta p_\parallel^{(2)} \rangle$.

Now we specify the electron–ion interaction. In the following we consider the regularized screened potential $U(\mathbf{r}) = U_R(\mathbf{r})$ introduced in Sec. II A with

$$U_R(k_\parallel, k_\perp) = \frac{2}{(2\pi)^2} \left(\frac{1}{k_\perp^2 + \kappa^2} - \frac{1}{k_\perp^2 + \chi^2} \right), \tag{52}$$

where $\kappa^2 = k_\parallel^2 + \lambda^{-2}$, $\chi^2 = k_\parallel^2 + d^{-2}$ and $d^{-1} = \lambda^{-1} + \lambdabar^{-1}$. Carrying out the calculation of the $k_\perp$-integrals in Eqs. (50) and (51) (see, e.g., [20]) with the potential (52) and substituting into Eq. (49) for the regularized screened interaction we obtain

$$
\begin{aligned}
\langle \Delta E_{i\parallel}^{(2)} \rangle &= \frac{4Z^2 e^4 v_{i\parallel}}{m\delta^2 v_{r\parallel}^3} \sum_{n=1}^{\infty} n^2 \Bigg\{ 3\left[u_n\left(\kappa_n a, \kappa_n s\right) - u_n\left(\chi_n a, \chi_n s\right) \right]^2 \\
&+ \frac{2n^2}{\delta^2} \left[u_n\left(\kappa_n a, \kappa_n s\right) - u_n\left(\chi_n a, \chi_n s\right) \right] \left[\frac{1}{\kappa_n^2} T_n\left(\kappa_n a, \kappa_n s\right) - \frac{1}{\chi_n^2} T_n\left(\chi_n a, \chi_n s\right) \right] \\
&+ 2\delta^2 \left[\kappa_n^2 Q_n\left(\kappa_n a, \kappa_n s\right) + \chi_n^2 Q_n\left(\chi_n a, \chi_n s\right) - 2\kappa_n \chi_n D_n\left(\kappa_n a, \kappa_n s; \chi_n a, \chi_n s\right) \right] \Bigg\}.
\end{aligned}
\tag{53}
$$

Here

$$\kappa_n^2 = \frac{n^2}{\delta^2} + \frac{1}{\lambda^2}, \quad \chi_n^2 = \frac{n^2}{\delta^2} + \frac{1}{d^2}, \tag{54}$$

$$
\begin{aligned}
u_n(x, y) &= I_n(\xi) K_n(\eta), \\
T_n(x, y) &= \xi I_n'(\xi) K_n(\eta) + \eta K_n'(\eta) I_n(\xi), \\
Q_n(x, y) &= I_n(\xi) K_n(\eta) \left[\frac{1}{\xi} I_n'(\xi) K_n(\eta) + \frac{1}{\eta} K_n'(\eta) I_n(\xi) \right]
\end{aligned}
\tag{55}
$$

with $\xi = \min(x, y)$, $\eta = \max(x, y)$, and the modified Bessel functions I_n and K_n,

$$
\begin{aligned}
D_n\left(x, y; X, Y\right) &= \frac{1}{4n} \left[s_{n-1}\left(x, y\right) s_{n-1}\left(X, Y\right) - s_n\left(x, y\right) s_n\left(X, Y\right) \right. \\
&+ \left. s_{n-1}\left(y, x\right) s_{n-1}\left(Y, X\right) - s_n\left(y, x\right) s_n\left(Y, X\right) \right],
\end{aligned}
\tag{56}
$$

$$
s_n(x, y) = \begin{cases} I_n\left(x\right) K_{n+1}\left(y\right) & y > x \\ -I_{n+1}\left(y\right) K_n\left(x\right) & y < x \\ \frac{1}{2}\left[I_n\left(x\right) K_{n+1}\left(x\right) - I_{n+1}\left(x\right) K_n\left(x\right) \right] & y = x \end{cases}.
\tag{57}
$$

For a study of the convergence of the series in Eq. (53) we note that in all terms the modified Bessel functions I_n carry the smaller argument $\propto \min(a, s)$, while the K_n depend on $\max(a, s)$. At large harmonic numbers n both the indices and the arguments of these functions are large, and they behave as $I_n(n\xi)$, $K_n(n\eta)$. Therefore the case $s = a$ is most critical for the convergence of (53). This is intuitively clear as the gyrating electron hits the ion on such a trajectory. This should not matter for the potential (52), which has been regularized near the origin for exactly that purpose. Since at large $n \to \infty$ and at $s = a$ the summand in Eq. (53) involves the functions $I_n(nx)$ and $K_n(nx)$ and their derivatives the further analysis can be done using the uniform asymptotic expansions of the modified Bessel functions [20, 21]. Insertion of the expansions shows indeed that the nth member of the series are of the order $O(n^{-4})$, so the series converges even for $s = a$. On the other hand, the energy transfer for the unregularized potentials U_C and U_D diverges for $s = a$.

For the screened but unregularized potential, i.e. in the limit $\lambdabar \to 0$, all functions u_n, T_n, Q_n and D_n involving χ_n in their arguments tend to zero. There remains

$$
\begin{aligned}
\langle \Delta E_{i\parallel}^{(2)} \rangle &= \frac{4Z^2 e^4 v_{i\parallel}}{m\delta^2 v_{r\parallel}^3} \sum_{n=1}^{\infty} n^2 \Big[3u_n^2\left(\kappa_n a, \kappa_n s\right) \\
&+ \frac{2n^2}{(\kappa_n \delta)^2} u_n\left(\kappa_n a, \kappa_n s\right) T_n\left(\kappa_n a, \kappa_n s\right) + 2(\kappa_n \delta)^2 Q_n\left(\kappa_n a, \kappa_n s\right) \Big].
\end{aligned}
\tag{58}
$$

Now insertion of the uniform expansions of the modified Bessel functions shows that the members of this series are independent of n for $n \to \infty$ and $s = a$, hence the series diverges. In the Coulomb case, i.e. for $\lambda \to \infty$, $\kappa_n = n/\delta$, the resulting series are geometric and can be summed in closed form

$$\langle \Delta E_{i\parallel}^{(2)} \rangle \simeq \frac{Z^2 \not{e}^4 v_{i\parallel}}{2m\delta^2 v_{r\parallel}^3} \frac{1}{\sqrt{(1+\xi^2)(1+\eta^2)}} \frac{1}{\sinh \Psi(\xi,\eta)} \tag{59}$$

$$\times \left\{ [2 + \Xi_1(\xi,\eta)] e^{-\Psi(\xi,\eta)} + \frac{\Xi_2(\xi,\eta)}{\sinh \Psi(\xi,\eta)} \right\}$$

with $\xi = \min\left(\frac{a}{\delta}, \frac{s}{\delta}\right)$, $\eta = \max\left(\frac{a}{\delta}, \frac{s}{\delta}\right)$, $\Psi(\xi,\eta) = \varphi(\xi) - \varphi(\eta)$ and

$$\varphi(\xi) = \sqrt{\xi^2 + 1} - \ln\left(\frac{\sqrt{\xi^2 + 1} + 1}{\xi}\right), \tag{60}$$

$$\Xi_1(\xi,\eta) = \frac{5(1+\xi^2)^{3/2}}{6\xi^2(1+\eta^2)^{3/2}} + \frac{5(1+\eta^2)^{3/2}}{6\eta^2(1+\xi^2)^{3/2}} - \frac{1}{3}\left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right)$$

$$- \frac{(1+\eta^2)^{3/2}}{2\eta^2(1+\xi^2)^{1/2}} - \frac{(1+\xi^2)^{3/2}}{2\xi^2(1+\eta^2)^{1/2}}, \tag{61}$$

$$\Xi_2(\xi,\eta) = \frac{(1+\xi^2)^{3/2}}{\xi^2} - \frac{(1+\eta^2)^{3/2}}{\eta^2}.$$

For the limit $|s - a| \to 0$ the Taylor expansion of the function $\Psi(\xi,\eta)$ is used. This yields

$$\langle \Delta E_{i\parallel}^{(2)} \rangle \simeq \frac{Z^2 \not{e}^4 v_{i\parallel}}{m\delta v_{r\parallel}^3} \frac{1}{\eta(1+\eta^2)^{3/2}} \frac{1}{|s-a|} \tag{62}$$

which exhibits the divergence at $s = a$. Note that in Eq. (62) $\eta = a/\delta = v_{e\perp}/|v_{r\parallel}|$ and does not depend on the strength of the magnetic field.

For later purposes we also note the limits of Eq. (58) for a small electron transversal velocity with $a \ll \delta$,

$$\langle \Delta E_{i\parallel}^{(2)} \rangle \simeq \left(\frac{Z\not{e}^2}{s}\right)^2 \frac{2v_{i\parallel}}{mv_{r\parallel}^3} [(\kappa_1 s) K_1(\kappa_1 s)]^2 \left[1 + \frac{a^2}{\delta^2} \mathfrak{F}(\kappa_1\delta, \kappa_1 s, \kappa_2 s)\right] \tag{63}$$

with

$$\mathfrak{F}(\zeta, \lambda, \mu) = \frac{3}{2} + \frac{\zeta^2}{2}\left[1 + \frac{2K_1'(\lambda)}{\lambda K_1(\lambda)} + \left(\frac{\mu^2 K_2(\mu)}{\lambda^2 K_1(\lambda)}\right)^2\right] + \frac{1}{\zeta^2}\left[1 + \frac{\lambda K_1'(\lambda)}{K_1(\lambda)}\right]. \tag{64}$$

Because of the symmetry of Eq. (58) in respect to its arguments the limit of a small impact parameter $s \ll \delta$, is given by Eqs. (63) and (64) with the roles of a and s interchanged. Similarly from Eq. (53) in the case of the regularized potential and for vanishing cyclotron radius ($v_{e\perp} = 0$) we obtain

$$\langle \Delta E_{i\parallel}^{(2)} \rangle = \left(\frac{Z\not{e}^2}{s}\right)^2 \frac{2v_{i\parallel}}{mv_{r\parallel}^3} [(\kappa_1 s) K_1(\kappa_1 s) - (\chi_1 s) K_1(\chi_1 s)]^2. \tag{65}$$

Comparing the first term in Eq. (63) with Eq. (65) we conclude that the additional modified Bessel function in Eq. (65) with the argument $\chi_1 s$ guarantees the convergence of the energy transfer at small impact parameter s.

For the practical applications and for general interaction potential in Appendix A we also performe the s-integration of the second order energy transfer, Eq. (49).

V. ARBITRARY ION MOTION IN A STRONG MAGNETIC FIELD

After the discussion of the energy loss for an ion moving parallel to a magnetic field of arbitrary strength we return to the general case, where the ion velocity has a component transverse to the field. As mentioned above the integrations

and summation in Eq. (46) cannot be done in this case unless other simplifications are made. In the following we consider strong magnetic fields and in Eq. (46) keep only the terms with $n = 0, \pm 1$ and expand the remaining Bessel functions $J_0(k_\perp a) \simeq 1 - (k_\perp a)^2/4$, $J_1(k_\perp a) \simeq k_\perp a/2$ with respect to the cyclotron radius of electrons. Note that this approximation can be alternatively formulated as a smallness of the electron transverse velocity $v_{e\perp}$. We obtain two contributions to the energy transfer, $\langle \Delta E_i^{(2)} \rangle_\text{I}$ which is independent of the cyclotron radius and $\langle \Delta E_i^{(2)} \rangle_\text{II}$ which is proportional to a^2. We split $\langle \Delta E_i^{(2)} \rangle_\text{I}$ according to

$$\langle \Delta E_i^{(2)} \rangle_\text{I} = \langle \Delta E_i^{(2)} \rangle_\text{I1} + \langle \Delta E_i^{(2)} \rangle_\text{I2}, \tag{66}$$

where

$$\begin{aligned}
\langle \Delta E_i^{(2)} \rangle_\text{I1} &= \frac{(2\pi)^2 Z^2 e^4}{2m} \int d\mathbf{k} d\mathbf{k}' U(\mathbf{k}) U(\mathbf{k}') (\mathbf{k} \cdot \mathbf{v}_i) (\mathbf{k} \cdot \mathbf{b})(\mathbf{k}' \cdot \mathbf{b}) J_0(Qs) \\
&\quad \times \, \delta((\mathbf{k} + \mathbf{k}') \cdot \mathbf{v}_r) \delta'(\mathbf{k} \cdot \mathbf{v}_r),
\end{aligned} \tag{67}$$

$$\begin{aligned}
\langle \Delta E_i^{(2)} \rangle_\text{I2} &= \frac{2\pi Z^2 e^4}{m\omega_c} \int d\mathbf{k} d\mathbf{k}' U(\mathbf{k}) U(\mathbf{k}') (\mathbf{k} \cdot \mathbf{v}_i) J_0(Qs) \delta((\mathbf{k} + \mathbf{k}') \cdot \mathbf{v}_r) \\
&\quad \times \left[g_3(\mathbf{k}, \mathbf{k}') \left(\frac{1}{\mathbf{k} \cdot \mathbf{v}_r} - \frac{1}{\mathbf{k} \cdot \mathbf{v}_r - \omega_c} \right) + \pi g_2(\mathbf{k}, \mathbf{k}') \delta(\mathbf{k} \cdot \mathbf{v}_r - \omega_c) \right].
\end{aligned} \tag{68}$$

As in Sec. IV we will assume axially symmetric potentials $U(\mathbf{k}) = U(|k_\parallel|, k_\perp)$. The integration is done by using cylindrical coordinates oriented along $\mathbf{n}_r = \mathbf{v}_r/v_r$, i.e. any vector $\mathbf{C}$ will be represented as $\mathbf{C} = C_\parallel^{(r)} \mathbf{n}_r + \mathbf{C}_\perp^{(r)}$. For the Bessel functions we use the addition theorem [20]. The angular integrals are trivial as they involve powers of trigonometric functions. Note that the contribution of the term proportional to the function $g_3(\mathbf{k}, \mathbf{k}')$ in Eq. (68) vanishes due to the antisymmetrical behavior of this function with respect to the azymuthal angles of $\mathbf{k}_\perp^{(r)}$ and $\mathbf{k}'_\perp^{(r)}$. Then for the energy transfers after straightforward calculations we obtain

$$\langle \Delta E_i^{(2)} \rangle_\text{I1} = \frac{Z^2 e^4 v_{i\perp}^2}{m v_r^6} \left[(v_{e\parallel}^2 - v_i^2) T_{12}^2(s) + v_{e\parallel} v_{r\parallel} T_{01}(s) T_{03}(s) \right], \tag{69}$$

$$\begin{aligned}
\langle \Delta E_i^{(2)} \rangle_\text{I2} &= \frac{Z^2 e^4}{m v_r^6} \Bigg\{ v_{i\perp}^2 \left[2q^2 \left(v_{e\parallel} v_{i\parallel} - v_i^2 \right) \mathcal{T}_{01}^2(q, s) - v_{e\parallel} v_{r\parallel} \mathcal{T}_{01}(q, s) \mathcal{T}_{03}(q, s) \right] \\
&\quad + \left[2v_r^2(v_{e\parallel} v_{i\parallel} - v_i^2) - v_{i\perp}^2(v_{e\parallel}^2 - v_i^2) \right] \mathcal{T}_{12}^2(q, s) \Bigg\},
\end{aligned} \tag{70}$$

where $q = \delta^{-1} = \omega_c/v_r$, $T_{\nu\mu}(s) = \mathcal{T}_{\nu\mu}(0, s)$ and

$$\mathcal{T}_{\nu\mu}(q, s) = \frac{(2\pi)^2}{2} \int_0^\infty U(q, k_\perp) J_\nu(k_\perp s) k_\perp^\mu dk_\perp. \tag{71}$$

We now specify the interaction potential and explicitly calculate the functions (71). From Eqs. (52) and (71) we obtain in the regularized and screened case [20]

$$\begin{aligned}
\mathcal{T}_{12}^\text{R}(q, s) &= \kappa_1 K_1(\kappa_1 s) - \chi_1 K_1(\chi_1 s), \\
\mathcal{T}_{03}^\text{R}(q, s) &= \chi_1^2 K_0(\chi_1 s) - \kappa_1^2 K_0(\kappa_1 s), \\
\mathcal{T}_{01}^\text{R}(q, s) &= K_0(\kappa_1 s) - K_0(\chi_1 s)
\end{aligned} \tag{72}$$

with $\kappa_1^2 = q^2 + \lambda^{-2}$ and $\chi_1^2 = q^2 + d^{-2}$. The functions $T_{12}^\text{R}(s)$, $T_{03}^\text{R}(s)$ and $T_{01}^\text{R}(s)$ are easily obtained from Eq. (72) setting there $q = 0$, i.e. $\kappa_1 = 1/\lambda$ and $\chi_1 = 1/d$. We investigate the asymptotic behavior of the functions in Eq. (72) in the limit $s \to 0$. As $K_0(z) \sim \ln(1/z)$ and $K_1(z) \sim 1/z$ the divergence is not worse than logarithmic and will cause no harm when integrating over the impact parameter s. This is not so in the case of the screened potential. Insertion of Eq. (72) at $\lambda \to 0$ (i.e. $\chi_1 \to \infty$) into Eq. (69) yields

$$\langle \Delta E_i^{(2)} \rangle_\text{I1} = \left(\frac{Z e^2}{s} \right)^2 \frac{v_{i\perp}^2}{m v_r^6} \left\{ (v_{e\parallel}^2 - v_i^2) [\rho K_1(\rho)]^2 - v_{e\parallel} v_{r\parallel} [\rho K_0(\rho)]^2 \right\} \tag{73}$$

with $\rho = s/\lambda$, which behaves like s^{-2} for small impact parameters. On the other hand all functions in Eq. (72) vanish exponentially for large impact parameters because of the finite range of the potentials U_{R} and U_{D}. We obtain the Coulomb case by taking the limit $\lambda \to \infty$ in Eq. (73). This yields

$$\langle \Delta E_i^{(2)} \rangle_{\mathrm{I1}} = \left(\frac{Ze^2}{s} \right)^2 \frac{v_{i\perp}^2}{mv_r^6} (v_{e\parallel}^2 - v_i^2) \tag{74}$$

which is precisely the energy transfer for tight helices [16].

We turn now to the next term $\langle \Delta E_i^{(2)} \rangle_{\mathrm{I2}}$ in Eq. (66), i.e. Eq. (70) with functions from Eq. (72). For the screened potential U_{D} the insertion of Eq. (72) at $\lambda \to 0$ yields

$$\langle \Delta E_i^{(2)} \rangle_{\mathrm{I2}} = \left(\frac{Ze^2}{s} \right)^2 \frac{1}{mv_r^6} \left\{ v_{i\perp}^2 \left[\frac{2}{(\kappa_1\delta)^2} (v_{e\parallel}v_{i\parallel} - v_i^2) + v_{e\parallel}v_{r\parallel} \right] [\rho_1 K_0(\rho_1)]^2 \tag{75} \right.$$
$$\left. + \left[2v_r^2(v_{e\parallel}v_{i\parallel} - v_i^2) - v_{i\perp}^2(v_{e\parallel}^2 - v_i^2) \right] [\rho_1 K_1(\rho_1)]^2 \right\}$$

with $\rho_1 = \kappa_1 s$. For parallel ion motion, $v_{i\perp} \to 0$, the first term vanishes $\langle \Delta E_i^{(2)} \rangle_{\mathrm{I1}} = 0$ and there remains a contribution to $\langle \Delta E_i^{(2)} \rangle_{\mathrm{I2}}$, which as expected is equal to the leading term of the corresponding expansion of Eq. (63) in orders of a/δ.

Taking now the limit $\lambda \to \infty$ for the unscreened Coulomb interaction we see that $\langle \Delta E_i^{(2)} \rangle_{\mathrm{I2}}$ vanishes exponentially for $s > \delta$. Hence the energy transfer is given by the tight helix term Eq. (74). For the case $s < \delta$ and hence $\rho_1 \simeq 0$ we obtain

$$\langle \Delta E_i^{(2)} \rangle_{\mathrm{I1}} + \langle \Delta E_i^{(2)} \rangle_{\mathrm{I2}} = \left(\frac{Ze^2}{s} \right)^2 \frac{2\mathbf{v}_i \cdot \mathbf{v}_r}{mv_r^4}. \tag{76}$$

This is just the stretched helix case considered in Ref. [16]. Equation (76) is the same as the second-order energy transfer in a field-free case (see, e.g., Refs. [11]) but the full relative velocity $\mathbf{u}_r = \mathbf{v}_e - \mathbf{v}_i$ is replaced here by the relative velocity of the electron guiding center $\mathbf{v}_r$.

These results are obtained in the limit $a \to 0$ where the electrons move along their guiding center trajectories. Moreover, for $\omega_c \to \infty$ also the pitch $\delta \to 0$ and these trajectories are rectilinear along the lines of the magnetic field and the energy transfer is given by $\langle \Delta E_i^{(2)} \rangle_{\mathrm{I1}}$. For a finite ω_c corresponding to a finite pitch the contribution $\langle \Delta E_i^{(2)} \rangle_{\mathrm{I2}}$ describes the perturbation of the guiding center trajectory.

The quadratic term $\langle \Delta E_i^{(2)} \rangle_{\mathrm{II}} \sim a^2$ accounts for the finite cyclotron motion of the electrons. In general this term is obtained from Eq. (46) and reads

$$\langle \Delta E_i^{(2)} \rangle_{\mathrm{II}} = -\frac{\pi i Z^2 e^4}{4m} a^2 \int d\mathbf{k} d\mathbf{k}' U(\mathbf{k}) U(\mathbf{k}') (\mathbf{k} \cdot \mathbf{v}_i) J_0(Qs) \delta((\mathbf{k} + \mathbf{k}') \cdot \mathbf{v}_r) \tag{77}$$
$$\times \left\{ (k_\perp^2 + k_\perp'^2) G_0(\mathbf{k}, \mathbf{k}') + k_\perp k_\perp' \left[e^{i(\theta - \theta')} G_1(\mathbf{k}, \mathbf{k}') + e^{-i(\theta - \theta')} G_{-1}(\mathbf{k}, \mathbf{k}') \right] \right\}.$$

Here $G_0(\mathbf{k}, \mathbf{k}')$ and $G_{\pm 1}(\mathbf{k}, \mathbf{k}')$ are given by Eq. (44). Using the same techniques as before the straightforward calculation yields

$$\langle \Delta E_i^{(2)} \rangle_{\mathrm{II}} = \left(\frac{Ze^2 a}{s^2} \right)^2 \frac{1}{2mv_r^2} (\mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2). \tag{78}$$

Here we restrict ourselves to give the result for the screened potential U_{D}

$$\mathcal{Q}_0 = \frac{v_{e\parallel}v_{r\parallel}v_{i\perp}^2}{v_r^4} \left(\frac{v_{i\perp}^2}{8v_r^2} - \frac{v_{r\parallel}^2}{v_r^2} \right) [\rho^2 K_0(\rho)]^2 + \frac{v_{i\perp}^2}{v_r^2} \left[\frac{v_{e\parallel}v_{i\parallel} - v_i^2}{v_r^2} \left(1 - \frac{3v_{i\perp}^2}{4v_r^2} \right) \right. \tag{79}$$
$$\left. + \frac{v_{e\parallel}v_{r\parallel}}{v_r^2} \left(1 - \frac{3v_{i\perp}^2}{2v_r^2} \right) \right] [\rho^2 K_1(\rho)]^2 + \frac{3v_{e\parallel}v_{r\parallel}v_{i\perp}^4}{8v_r^6} [\rho^2 K_2(\rho)]^2,$$

$$
\begin{aligned}
\mathcal{Q}_1 = & \left\{ \frac{v_{e\|}v_{r\|}v_{i\perp}^2}{v_r^4}\left(1 - \frac{17v_{i\perp}^2}{8v_r^2}\right) + \frac{v_{r\|}^2\left(v_{e\|}v_{i\|} - v_i^2\right)}{v_r^4}\left(1 + \frac{v_{i\perp}^2}{2v_r^2}\right) \right. \\
& + \frac{q^2}{\kappa_1^2}\frac{v_{i\perp}^2}{v_r^2}\left[\frac{v_{e\|}v_{r\|}}{v_r^2}\left(5 - \frac{12v_{i\perp}^2}{v_r^2}\right) + \frac{v_{e\|}v_{i\|} - v_i^2}{v_r^2}\left(10 - \frac{11v_{i\perp}^2}{v_r^2}\right)\right] \\
& + \left. \frac{2q^4}{\kappa_1^4}\frac{v_{i\perp}^2}{v_r^2}\left[\frac{v_{e\|}v_{r\|}}{v_r^2}\left(1 - \frac{2v_{i\perp}^2}{v_r^2}\right) + \frac{v_{e\|}v_{i\|} - v_i^2}{v_r^2}\left(7 - \frac{8v_{i\perp}^2}{v_r^2}\right)\right] \right\}\left[\rho_1^2 K_0(\rho_1)\right]^2 \\
& + \left\{ -\frac{3v_{e\|}v_{r\|}^3 v_{i\perp}^2}{v_r^6} + \frac{v_{e\|}v_{i\|} - v_i^2}{v_r^2}\left(\frac{3v_{i\perp}^4}{4v_r^4} + \frac{2v_{r\|}^2}{v_r^2}\right) \right. \\
& + \left. \frac{q^2}{\kappa_1^2}\left[\frac{v_{e\|}v_{r\|}v_{i\perp}^2}{v_r^4}\left(\frac{13v_{i\perp}^2}{v_r^2} - 7\right) + \frac{v_{e\|}v_{i\|} - v_i^2}{v_r^2}\left(\frac{16v_{i\perp}^4}{v_r^4} - \frac{23v_{i\perp}^2}{v_r^2} + 6\right)\right] \right\}\left[\rho_1^2 K_1(\rho_1)\right]^2 \\
& + \left[\frac{v_{e\|}v_{r\|}v_{i\perp}^2}{2v_r^4}\left(4 - \frac{7v_{i\perp}^2}{4v_r^2}\right) - \frac{v_{e\|}v_{i\|} - v_i^2}{v_r^2}\left(1 - \frac{3v_{i\perp}^2}{2v_r^2} + \frac{v_{i\perp}^4}{4v_r^4}\right)\right]\left[\rho_1^2 K_2(\rho_1)\right]^2 \\
& - \frac{q^2}{\kappa_1^2}\frac{v_{e\|}v_{i\|} - v_i^2}{v_r^2}\left[\frac{v_{i\perp}^2}{v_r^2}\left(2 - \frac{3v_{i\perp}^2}{2v_r^2}\right) + \frac{2q^2}{\kappa_1^2}\left(2 - \frac{3v_{i\perp}^2}{v_r^2} + \frac{2v_{i\perp}^4}{v_r^4}\right) + \frac{4q^4}{\kappa_1^4}\frac{v_{r\|}^2 v_{i\perp}^2}{v_r^4}\right]\rho_1^5 K_0(\rho_1)K_1(\rho_1),
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
\mathcal{Q}_2 = & \frac{v_{i\perp}^4}{v_r^4}\left(1 + \frac{8q^2}{\kappa_2^2}\right)\left[\frac{v_{e\|}v_{r\|}}{v_r^2} + \frac{v_{e\|}v_{i\|} - v_i^2}{2v_r^2}\left(1 + \frac{8q^2}{\kappa_2^2}\right)\right]\left[\rho_2^2 K_0(\rho_2)\right]^2 + \frac{2v_{i\perp}^2}{v_r^2}\left\{\frac{v_{e\|}v_{r\|}}{v_r^2} \right. \\
& \times \left. \left(1 - \frac{3v_{i\perp}^2}{4v_r^2}\right) + \frac{4q^2}{\kappa_2^2}\left[\frac{2\left(v_{e\|}v_{i\|} - v_i^2\right)}{v_r^2}\left(1 + \frac{v_{r\|}^2}{v_r^2}\right) - \frac{v_{e\|}v_{r\|}v_{i\perp}^2}{v_r^4}\right]\right\}\left[\rho_2^2 K_1(\rho_2)\right]^2 \\
& + \left[\frac{v_{e\|}v_{i\|} - v_i^2}{v_r^2}\left(\frac{2v_{r\|}^2}{v_r^2} + \frac{v_{i\perp}^4}{4v_r^4}\right) + \frac{v_{r\|}v_{e\|}v_{i\perp}^2}{v_r^4}\left(\frac{v_{i\perp}^2}{2v_r^2} - 2\right)\right]\left[\rho_2^2 K_2(\rho_2)\right]^2.
\end{aligned}
\tag{81}
$$

Here $\kappa_2^2 = 4/\delta^2 + 1/\lambda^2$, $\rho_2 = \kappa_2 s$. The cyclotron motion and the drift of the guiding center of the electron are coupled to each other. Therefore the perturbation of the cyclotron motion causes an additional perturbation of the guiding center motion. This effect is given by the first term $\mathcal{Q}_0$ in Eq. (78) which depends on magnetic field through the cyclotron radius a in the prefactor, while the arguments of the modified Bessel functions in Eq. (79) do not depend on magnetic field. In the other terms $\mathcal{Q}_1$ and $\mathcal{Q}_2$ the arguments of the Bessel functions ρ_1 and ρ_2 correspond to the first and second cyclotron harmonic perturbations, respectively.

For $v_{i\perp} = 0$ we have $\mathcal{Q}_0 = 0$ and

$$
\mathcal{Q}_1 = \frac{v_{i\|}}{v_{r\|}}\left\{\rho_1^4\left[K_0^2(\rho_1) - K_2^2(\rho_1)\right] + \left[2 + \frac{6}{(\kappa_1\delta)^2}\right]\rho_1^4 K_1^2(\rho_1) - \frac{4}{(\kappa_1\delta)^4}\rho_1^5 K_0(\rho_1)K_1(\rho_1)\right\},
\tag{82}
$$

$$
\mathcal{Q}_2 = \frac{2v_{i\|}}{v_{r\|}}\left[\rho_2^2 K_2(\rho_2)\right]^2.
\tag{83}
$$

With the help of the recursion relations of the modified Bessel functions it is easy to see that the resulting energy transfer $\langle \Delta E_i^{(2)}\rangle_{\mathrm{II}}$ agrees with the corresponding $\mathrm{O}(a^2/\delta^2)$-term of the energy transfer, Eq. (63), where the limit of parallel ion motion $v_{i\perp} \to 0$ was taken before the limit $a \ll \delta$.

VI. DISCUSSION AND CONCLUSION

In this paper, we have presented a detailed theoretical investigation of the energy transfer of a uniformly moving heavy ion due to the binary collision (BC) with the magnetized electrons. The BC energy transfer can only be evaluated explicitly in closed form in the limiting cases of a vanishing and an infinitely strong magnetic field. The BC treatment developed here is valid for arbitrary strengths of the magnetic field and arbitrary shapes of the interaction potential up to second order in the interaction strength. The purpose of this work was to investigate the ion energy transfer for finite magnetic fields which is explicitly calculated for a regularized and screened potential which is both of

finite range and less singular than the Coulomb interaction at the origin and as the limiting cases involves the Debye (i.e., screened) and Coulomb potentials. Two particular cases have been considered in detail: (i) Ion motion parallel to the magnetic field with an arbitrary strength. The energy transfer involves all harmonics of the electron cyclotron motion. (ii) The ion arbitrary motion with respect to the strong magnetic field when the electron cyclotron radius is much smaller than other characteristic length scales (e.g., screening length, pitch of electron helix etc.). We show that in the latter case the energy transfer receives two contributions which are responsible for the electron guiding center and cyclotron orbit perturbations.

We would like to mention that our current results leave still some questions open. It is clear that for the validity of the second–order perturbation BC theory developed here more critical are the domains of the small relative velocities v_r and/or impact parameters s. Moreover, for the binary electron–ion collisions in a magnetic field as given by the equation of motion (6) there are less integrals of motion than degrees of freedom which indicates the possibility of the chaotic dynamics in the system [12, 13, 14]. This immediately raises the question whether a perturbative treatment as proposed in this paper can be applied at all. We will address this issue in the fortcoming studies by showing some examples for the energy transfer obtained from a numerical solution of the equation of motion (6) and by making some comparison with the perturbative treatment. This topic is presently under investigation and will be published elsewhere.

Acknowledgments

APPENDIX A: INTEGRATED ENERGY TRANSFER FOR PARALLEL ION MOTION

The integration of the energy transfer Eq. (49) with Eqs. (50) and (51) with respect to the impact parameter s is facilitated by using the following relations for the Bessel functions

$$\int_0^\infty J_n\left(k_\perp s\right) J_n\left(k'_\perp s\right) s ds = \int_0^\infty J_{n-1}\left(k_\perp s\right) J_{n-1}\left(k'_\perp s\right) s ds = \frac{1}{k_\perp} \delta\left(k'_\perp - k_\perp\right). \tag{A1}$$

Using this relation we integrate the energy transfer Eq. (49) with respect to the impact parameter. This yields

$$\int_0^\infty \langle \Delta E_{i\parallel}^{(2)} \rangle s ds = \left(\frac{Z e^2}{\delta}\right)^2 \frac{4 v_{i\parallel}}{m v_{r\parallel}^3} \sum_{n=1}^\infty n^2 \left\{ 3\Phi_n\left(k_\parallel, a\right) + k_\parallel \frac{\partial}{\partial k_\parallel} \Phi_n\left(k_\parallel, a\right) \right. \tag{A2}$$
$$\left. + \frac{\delta^2}{2n} \left[\Psi_{n-1}\left(k_\parallel, a\right) - \Psi_{n+1}\left(k_\parallel, a\right) \right] \right\}_{k_\parallel = n/\delta},$$

where

$$\Phi_n\left(k_\parallel, a\right) = \int_0^\infty U_n\left(k_\parallel, a, s\right) s ds = \frac{(2\pi)^4}{4} \int_0^\infty U^2\left(k_\parallel, k_\perp\right) J_n^2\left(k_\perp a\right) k_\perp dk_\perp, \tag{A3}$$

$$\Psi_n\left(k_\parallel, a\right) = \int_0^\infty V_{n+1}\left(k_\parallel, s, a\right) s ds = \int_0^\infty V_n\left(k_\parallel, a, s\right) s ds \tag{A4}$$
$$= \frac{(2\pi)^4}{4} \int_0^\infty U^2\left(k_\parallel, k_\perp\right) J_n^2\left(k_\perp a\right) k_\perp^3 dk_\perp.$$

Using the recurrent relations between the Bessel functions we obtain

$$\Psi_{n-1}\left(k_\parallel, a\right) - \Psi_{n+1}\left(k_\parallel, a\right) = \frac{2n}{a} \frac{\partial}{\partial a} \Phi_n\left(k_\parallel, a\right). \tag{A5}$$

Thus the integrated energy transfer is expressed only by the functions Φ_n

$$\int_0^\infty \langle \Delta E_{i\parallel}^{(2)} \rangle s ds = \left(\frac{Z e^2}{\delta}\right)^2 \frac{4 v_{i\parallel}}{m v_{r\parallel}^3} \tag{A6}$$
$$\times \sum_{n=1}^\infty n^2 \left\{ 3\Phi_n\left(k_\parallel, a\right) + k_\parallel \frac{\partial}{\partial k_\parallel} \Phi_n\left(k_\parallel, a\right) + \frac{\delta^2}{a} \frac{\partial}{\partial a} \Phi_n\left(k_\parallel, a\right) \right\}_{k_\parallel = n/\delta}.$$

As an important particular case consider the regularized and screened interaction potential Eq. (52). For this potential Eq. (A3) for the function Φ_n yields [20]

$$\Phi_n \left(k_\parallel, a \right) = \frac{2}{\lambda^{-2} - d^{-2}} \left[I_n \left(\kappa a \right) K_n \left(\kappa a \right) - I_n \left(\chi a \right) K_n \left(\chi a \right) \right] - \frac{1}{2\kappa^2} H_n \left(\kappa a \right) - \frac{1}{2\chi^2} H_n \left(\chi a \right), \tag{A7}$$

where κ, χ and d have been introduced in Sec. IV and

$$H_n \left(\xi \right) = \xi \frac{\partial}{\partial \xi} \left[I_n \left(\xi \right) K_n \left(\xi \right) \right] = \xi \left[I_n' \left(\xi \right) K_n \left(\xi \right) + I_n \left(\xi \right) K_n' \left(\xi \right) \right]. \tag{A8}$$

Note that $H_n \left(\xi \right) = T_n \left(\xi, \xi \right)$, where $T_n \left(x, y \right)$ is defined in Eq. (55). From Eq. (A7) one can derive the function Φ_n for screened and Coulomb potentials. For the screened but unregularized potential (i.e. $\lambda \to 0$) all terms in Eq. (A7) containing χ vanish and the function Φ_n becomes

$$\Phi_n \left(k_\parallel, a \right) = -\frac{1}{2\kappa^2} H_n \left(\kappa a \right). \tag{A9}$$

In a limit $\lambda \to \infty$, i.e. in the case of unscreened Coulomb potential in Eq. (A9) the variable κ is replaced by $\left| k_\parallel \right|$.

APPENDIX B: INTEGRATED ENERGY TRANSFER FOR AN INFINITELY STRONG MAGNETIC FIELD

Consider the integrated energy transfer for an arbitrary ion motion and for infinitely strong magnetic field. The integration of Eq. (69) with respect to s involves two integrals of the functions $T_{12}^2 \left(s \right)$ and $T_{01} \left(s \right) T_{03} \left(s \right)$ which can be evaluated employing the relation (A1). In a general case with $q \neq 0$ we obtain

$$\mathcal{T} \left(q, \varkappa \right) = \int_0^\infty \mathcal{T}_{12}^2 \left(q, s \right) s ds = \int_0^\infty \mathcal{T}_{01} \left(q, s \right) \mathcal{T}_{03} \left(q, s \right) s ds = \frac{(2\pi)^4}{4} \int_0^\infty U^2 \left(q, k_\perp \right) k_\perp^3 dk_\perp. \tag{B1}$$

Obviously the integrations in Eq. (B1) require that the interaction potential must decay faster than r^{-1} at large distances and must increase slower than r^{-1} at small ones. In particular, for regularized screened potential from Eq. (B1) we find

$$\mathcal{T} \left(q, \varkappa \right) = \frac{2q^2\lambda^2 + \varkappa^2 + 1}{2 \left(\varkappa^2 - 1 \right)} \ln \frac{q^2\lambda^2 + \varkappa^2}{q^2\lambda^2 + 1} - 1 \tag{B2}$$

with $\varkappa = \lambda/d = 1 + \lambda/\lambdabar$ which at $q = 0$ is simplified to

$$\mathcal{U}_0(\varkappa) = \mathcal{T} \left(0, \varkappa \right) = \frac{\varkappa^2 + 1}{\varkappa^2 - 1} \ln \varkappa - 1. \tag{B3}$$

Thus the s-integrated energy transfer in the presence of an infinitely strong magnetic field reads

$$2\pi \int_0^\infty \langle \Delta E_i^{(2)} \rangle_{\text{I1}} s ds = \frac{2\pi Z^2 e^4 v_{i\perp}^2}{m v_r^6} \mathcal{U}_0 \left(\varkappa \right) \left(2v_{e\parallel}^2 - v_i^2 - v_{e\parallel} v_{i\parallel} \right). \tag{B4}$$

The quantity $\mathcal{U}_0 \left(\varkappa \right)$ in Eq. (B4) can be treated as a modified Coulomb logarithm.

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